SYMPLECTIC AUTOMORPHISMS OF PRIME ORDER ON K3 SURFACES

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ABSTRACT. We study algebraic K3 surfaces (defined over the complex number field) with a symplectic automorphism of prime order. In particular we consider the action of the automorphism on the second cohomology with integer coefficients (by a result of Nikulin this action is independent on the choice of the K3 surface). With the help of elliptic fibrations we determine the invariant sublattice and its perpendicular complement, and show that the latter coincides with the Coxeter-Todd lattice in the case of automorphism of order three.

0. INTRODUCTION

In the paper [Ni1] Nikulin studies finite abelian groups G acting symplectically (i.e. $G_{|H^{2,0}(X,\mathbb{C})} = id_{|H^{2,0}(X,\mathbb{C})}$) on K3 surfaces (defined over $\mathbb{C}$). One of his main result is that the action induced by G on the cohomology group $H^2(X,\mathbb{Z})$ is unique up to isometry. In [Ni1] all abelian finite groups of automorphisms of a K3 surface acting symplectically are classified. Later Mukai in [Mu] extends the study to the non abelian case. Here we consider only abelian groups of prime order p which, by Nikulin, are isomorphic to $\mathbb{Z}/p\mathbb{Z}$ for $p = 2, 3, 5, 7$.

In the case of $p = 2$ the group is generated by an involution, which is called by Morrison in [Mo, Def. 5.1] Nikulin involution. This was very much studied in the last years, in particular because of its relation with the Shioda-Inose structure (cf. e.g. [CD], [GL], [vGT], [L], [Mo]). In [Mo] Morrison proves that the isometry induced by a Nikulin involution ι on the lattice $\Lambda_{K3} \simeq U \oplus U \oplus U \oplus E_8(-1) \oplus E_8(-1)$, which is isometric to $H^2(X,\mathbb{Z})$, switches the two copies of $E_8(-1)$ and acts as the identity on the sublattice $U \oplus U \oplus U$. As a consequence one sees that $(H^2(X,\mathbb{Z})^{\iota^*})^\perp$ is the lattice $E_8(-2)$. This implies that the Picard number ρ of an algebraic K3 surface admitting a Nikulin involution is at least nine. In [vGS] van Geemen and Sarti show that if $\rho \geq 9$ and $E_8(-2) \subset NS(X)$ then the algebraic $K3$ surface X admits a Nikulin involution and they classify completely these $K3$ surfaces. Moreover they discuss many examples and in particular those surfaces admitting an elliptic fibration with a section of order two. This section operates by translation on the fibers and defines a Nikulin involution on the K3 surface.

The aim of this paper is to identify the action of a symplectic automorphism σ_p of the remaining possible prime orders $p = 3, 5, 7$ on the K3 lattice Λ_{K3} and to describe such algebraic K3 surfaces with minimal possible Picard number. Thanks to Nikulin's result ([Ni1, Theorem 4.7]), to find the action on Λ_{K3}, it suffices to identify the action in one special case. For this purpose it seemed to be convenient to study algebraic K3 surfaces with an elliptic fibration with a section of order three, five, resp. seven. Then the translation by this section is a symplectic automorphism of the surface of the same order. A

concrete analysis leads us to the main result of the paper which is the description of the lattices $H^2(X,\mathbb{Z})^{\sigma_p^*}$ and $\Omega_p = (H^2(X,\mathbb{Z})^{\sigma_p^*})^\perp$ given in the Theorem 4.1. The proof of the main theorem consists in the Propositions 4.2, 4.4, 4.6. We describe the lattice Ω_p also as $\mathbb{Z}[\omega_p]$-lattices, where ω_p is a primitive p root of the unity. This kind of lattices are studied e.g. in [Ba], [BS] and [E]. In particular in the case $p = 3$ the lattice Ω_3 is the Coxeter-Todd lattice with the form multiplied by -2, $K_{12}(-2)$, which is described in [CT] and in [CS]. The elliptic surfaces we used to find the lattices Ω_p do not have the minimal possible Picard number. We prove in Proposition 5.1 that for K3 surfaces, X, with minimal Picard number and symplectic automorphism, if L is a class in $NS(X)$ which is invariant for the automorphisms, with $L^2 = 2d > 0$, then either $NS(X) = \mathbb{Z}L \oplus \Omega_p$ or the latter is a sublattice of index p in $NS(X)$. Using this result and the one of the Proposition 5.2 we describe the coarse moduli space of the algebraic K3 surfaces admitting a symplectic automorphism of prime order.

The structure of the paper is the following: in section 1 we compute the number of moduli of algebraic K3 surfaces admitting a symplectic automorphism of order p and their minimal Picard number. In section 2 we give the definition of $\mathbb{Z}[\omega_p]$-lattice and we associate to it a module with a bilinear form, which in some cases is a $\mathbb{Z}$-lattice (we use this construction in section 4 to describe the lattices Ω_p as $\mathbb{Z}[\omega_p]$-lattices). In section 3 we recall some results about elliptic fibrations and elliptic K3 surfaces (see e.g. [Mi1], [Mi2], [Shim], [Shio] for more on elliptic K3 surfaces). In particular we introduce the three elliptic fibrations which we use in section 4 and give also their Weierstrass form. In section 4 we state and proof the main result, Theorem 4.1: we identify the lattices Ω_p and we describe them as $\mathbb{Z}[\omega_p]$-lattices. In section 5 we describe the Néron-Severi group of K3 surfaces admitting a symplectic automorphism and having minimal Picard number (Proposition 5.1). In section 5 we describe the coarse moduli space of the algebraic K3 surfaces admitting a symplectic automorphism and the Néron-Severi group of those having minimal Picard number.

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1. Preliminary results

Definition 1.1. *A symplectic automorphism σ_p of order p on a K3 surface X is an automorphism such that:*

1. the group G generated by σ_p is isomorphic to $\mathbb{Z}/p\mathbb{Z}$,
2. $\sigma_p^*(\delta) = \delta$, for all δ in $H^{2,0}(X)$.

We recall that by [Ni1] an automorphism on a K3 surface is symplectic if and only if it acts as the identity on the transcendental lattice T_X. In local coordinates at a fixed point σ_p has the form $\mathrm{diag}(\omega_p, \omega_p^{p-1})$ where ω_p is a primitive p-root of unity. By a result of Nikulin the only possible values for p are 2,3,5,7 see [Ni1, Theorem 4.5] and [Ni1, §5]. The automorphism σ_3 has six fixed points on X, σ_5 has four fixed points and σ_7 has three fixed points. The automorphism σ_p induces a σ_p^* isometry on $H^2(X,\mathbb{Z}) \cong \Lambda_{K3}$. Nikulin proved [Ni1, Theorem 4.7] that if σ_p is symplectic, then the action of σ_p^* is unique up to isometry of Λ_{K3}.

Let ω_p be a primitive p-root of the unity. The vector space $H^2(X,\mathbb{C})$ can be decomposed

in eigenspaces of the eigenvalues 1 and ω_p^i:

$$H^2(X,\mathbb{C}) = H^2(X,\mathbb{C})^{\sigma_p^*} \oplus \left(\bigoplus_{i=1,\ldots,p-1} H^2(X,\mathbb{C})_{\omega_p^i} \right).$$

We observe that the non rational eigenvalues ω_p^i have all the same multiplicity. So we put:
$a_p :=$ multiplicity of the eigenvalue 1, $b_p :=$ multiplicity of the eigenvalues ω_p^i.

In the following we find a_p and b_p by using the Lefschetz fixed point formula:

$$(1) \qquad \mu_p = \sum_r (-1)^r \mathrm{trace}(\sigma_p^* | H^r(X,\mathbb{C}))$$

where μ_p denotes the number of fixed points. For $K3$ surfaces we obtain

$$\mu_p = 1 + 0 + \mathrm{trace}(\sigma_p^* | H^2(X,\mathbb{C})) + 0 + 1.$$

Proposition 1.1. *Let X, σ_p, a_p, b_p be as above, $p = 3, 5, 7$. Let ρ_p be the Picard number of X, and let m_p be the dimension of the moduli space of the algebraic K3 surfaces admitting a symplectic automorphism of order p. Then*

$$\begin{array}{cccc}
a_3 = 10 & b_3 = 6 & \rho_3 \geq 13 & m_3 \leq 7 \\
a_5 = 6 & b_5 = 4 & \rho_5 \geq 17 & m_5 \leq 3 \\
a_7 = 4 & b_7 = 3 & \rho_7 \geq 19 & m_7 \leq 1.
\end{array}$$

Proof. The proof is similar in all the cases, here we give the details only in the case $p = 5$. A symplectic automorphism of order five on a K3 surface has exactly four fixed points. Applying the Lefschetz fixed points formula (1), we have $a_5 + b_5(\omega_5 + \omega_5^2 + \omega_5^3 + \omega_5^4) = 2$. Since $\omega_p^{p-1} = -(\sum_{i=0}^{p-2} \omega_p^i)$, the equation becomes $a_5 - b_5 = 2$.
Since $\dim H^2(X,\mathbb{C}) = 22$, a_5 and b_5 have to satisfy:

$$(2) \qquad \left\{ \begin{array}{rcl} a_5 - b_5 & = & 2 \\ a_5 + 4b_5 & = & 22. \end{array} \right.$$

We have

$\dim H^2(X,\mathbb{C})^{\sigma_5^*} = 6 = a_5$ and
$\dim H^2(X,\mathbb{C})_{\omega_5} = \dim H^2(X,\mathbb{C})_{\omega_5^2} = \dim H^2(X,\mathbb{C})_{\omega_5^3} = \dim H^2(X,\mathbb{C})_{\omega_5^4} = 4 = b_5.$

Since $T_X \otimes \mathbb{C} \subset H^2(X,\mathbb{C})^{\sigma_5^*}$, $(H^2(X,\mathbb{C})^{\sigma_5^*})^\perp = H^2(X,\mathbb{C})_{\omega_5} \oplus H^2(X,\mathbb{C})_{\omega_5^2} \oplus H^2(X,\mathbb{C})_{\omega_5^3} \oplus H^2(X,\mathbb{C})_{\omega_5^4} \subset NS(X) \otimes \mathbb{C}$. We consider only algebraic $K3$ surfaces and so we have an ample class h on X, by taking $h + \sigma_5^* h + \sigma_5^{*2} h + \sigma_5^{*3} h + \sigma_5^{*4} h$ we get a σ_5-invariant class, hence in $H^2(X,\mathbb{C})^{\sigma_5^*}$. From here it follows that $\rho_p = \mathrm{rank}\, NS(X) \geq 16 + 1 = 17$, whence rank $T_X \leq 22 - 17 = 5$. The number of moduli is at most $20 - 17 = 3$. $\qquad \square$

Remark. In [Ni1, §10] Nikulin computes $\mathrm{rank}(H^2(X,\mathbb{Z})^{\sigma_p^*})^\perp = (p-1)b_p$ and $\mathrm{rank}(H^2(X,\mathbb{Z})^{\sigma_p^*}) = a_p$. In [Ni1, Lemma 4.2] he also proves that there are no classes with self intersection -2 in the lattices $(H^2(X,\mathbb{Z})^{\sigma_p^*})^\perp$; we describe these lattices in the sections 4.1, 4.4, 4.6 and we find again the result of Nikulin.

2. THE $\mathbb{Z}[\omega]$-LATTICES

In the sections 4.2, 4.5, 4.7 our purpose is to describe $(H^2(X,\mathbb{Z})^{\sigma_p^*})^\perp$ as $\mathbb{Z}[\omega_p]$-lattice. We recall now some useful results on these lattices.

Definition 2.1. *Let p be an odd prime and $\omega := \omega_p$ be a primitive p-root of the unity. A $\mathbb{Z}[\omega]$-lattice is a free $\mathbb{Z}[\omega]$-module with an hermitian form (with values in $\mathbb{Z}[\omega]$). Its rank is its rank as $\mathbb{Z}[\omega]$-module.*

Let $\{L, h_L\}$ be a $\mathbb{Z}[\omega]$-lattice of rank n. The $\mathbb{Z}[\omega]$-module L is also a $\mathbb{Z}$-module of rank $(p-1)n$. In fact if e_i, $i = 1, \ldots, n$ is a basis of L as $\mathbb{Z}[\omega]$-module, $\omega^j e_i$, $i = 1, \ldots, n$, $j = 0, \ldots, p-2$ is a basis for L as $\mathbb{Z}$-module (recall that $\omega^{p-1} = -(\omega^{p-2} + \omega^{p-3} + \ldots + 1)$). The $\mathbb{Z}$-module L will be called $L_\mathbb{Z}$.

Let $\Gamma_p := Gal(\mathbb{Q}(\omega)/\mathbb{Q})$ be the group of the automorphisms of $\mathbb{Q}(\omega)$ which fix $\mathbb{Q}$. We recall that the group Γ_p has order $p-1$ and its elements are automorphisms ρ_i such that $\rho_i(1) = 1$, $\rho_i(\omega) = \omega^i$ where $i = 1, \ldots, p-1$. We define a bilinear form on $L_\mathbb{Z}$

$$(3) \qquad b_L(\alpha, \beta) = -\frac{1}{p} \sum_{\rho \in \Gamma_p} \rho(h_L(\alpha, \beta)).$$

Note that b_L takes values in $\frac{1}{p}\mathbb{Z}[\omega]$, so in general $\{L_\mathbb{Z}, b_L\}$ is not a $\mathbb{Z}$-lattice. We call it the *associated* module (resp. lattice) of the $\mathbb{Z}[\omega]$-lattice L.

Remark. Remark. By the definition of the bilinear form is clear that

$$b_L(\alpha, \beta) = -\frac{1}{p} Tr_{\mathbb{Q}(\omega)/\mathbb{Q}}(h_L(\alpha, \beta)).$$

For a precise definition of the Trace see [E, page 128]

2.1. **The $\mathbb{Z}$-lattice F_p.** We consider a K3 surface admitting an elliptic fibration. Let p be an odd prime number. Let I_p be a semistable fiber of a minimal elliptic fibration, i.e. (cf. section 3) I_p is a fiber which is a reducible curve, whose irreducible components are the edges of a p-polygon, as described in [Mi1, Table I.4.1], we denote the p-irreducible components by C_i, $i = 0, \ldots, p-1$, then

$$C_i \cdot C_j = \begin{cases} -2 & if \ i \equiv j \quad \mod p \\ 1 & if \ |i-j| \equiv 1 \quad \mod p \\ 0 & otherwise. \end{cases}$$

We consider now the free $\mathbb{Z}$-module F_p with basis the elements of the form $C_i - C_{i+1}$, $i = 1, \ldots, p-1$ and with bilinear form b_{F_p} which is the restriction of the intersection form to the basis $C_i - C_{i+1}$, then $\{F_p, b_{F_p}\}$ is a $\mathbb{Z}$-lattice.

2.2. **The $\mathbb{Z}[\omega_p]$-lattice G_p.** Let G_p be the $\mathbb{Z}[\omega]$-lattice $G_p := (1-\omega)^2\mathbb{Z}[\omega]$, with the standard hermitian form: $h(\alpha, \beta) = \alpha\bar{\beta}$. A basis for the $\mathbb{Z}$-module $G_{p,\mathbb{Z}}$ is $(1-\omega)^2\omega^i$, $i = 0, \ldots, p-2$.

On $\mathbb{Z}[\omega]$ we consider the bilinear form b_L defined in (3), with values in $\frac{1}{p}\mathbb{Z}$,

$$b(\alpha, \beta) = -\frac{1}{p} \sum_{\rho \in \Gamma_p} \rho(\alpha\bar{\beta}), \qquad \alpha, \beta \in \mathbb{Z}[\omega],$$

then we have

Lemma 2.1. *The bilinear form b restricted to G_p (denoted by b_G) has values in $\mathbb{Z}$ and coincides with the intersection form on F_p by using the map $F_p \to G_p$ defined by $C_i - C_{i+1} \mapsto \omega^i(1-\omega)^2$, $i = 1, \ldots, p-1$, $C_p = C_0$.*

Proof. An easy computation shows that we have for $p > 3$:

$$b_G(\omega^k(1-\omega)^2, \omega^h(1-\omega)^2) = \begin{cases} -6 & \text{if } k \equiv h \mod p, \\ 4 & \text{if } |k-h| \equiv 1 \mod p, \\ -1 & \text{if } |k-h| \equiv 2 \mod p, \\ 0 & \text{otherwise}, \end{cases}$$

and for $p = 3$:

$$b_G(\omega^k(1-\omega)^2, \omega^h(1-\omega)^2) = \begin{cases} -6 & \text{if } k \equiv h \mod p, \\ 3 & \text{if } |k-h| \equiv 1 \mod p. \end{cases}$$

The intersection form on F_p is easy to compute (cf. section 2.1) and this computation proves that the map $F_p \to G_p$ defined in the lemma is an isometry. $\qquad\square$

In section 4 we apply the results of this section and we find a $\mathbb{Z}[\omega]$-lattice $\{L_p, h_{L_p}\}$ such that

- $\{L_p, h_{L_p}\}$ contains G_p as sublattice;
- $\{L_{p,\mathbb{Z}}, b_{L_p}\}$ is a $\mathbb{Z}$-lattice;
- the $\mathbb{Z}$-lattice $\{L_{p,\mathbb{Z}}, b_{L_p}\}$ is isometric to the $\mathbb{Z}$-lattice $(H^2(X,\mathbb{Z})^{\sigma_p^*})^\perp$ for $p = 3, 5, 7$.

3. SOME GENERAL FACTS ON ELLIPTIC FIBRATIONS

In the next section we give explicit examples of K3 surfaces admitting a symplectic automorphism σ_p by using elliptic fibrations. Here we recall some general results about these fibrations.

Let X be an elliptic $K3$ surface, this means that we have a morphism

$$f : X \longrightarrow \mathbb{P}^1$$

such that the generic fiber is a (smooth) elliptic curve. We assume moreover that we have a section $s : \mathbb{P}^1 \longrightarrow X$. The sections of X generate the Mordell-Weil group MW_f of X and we take s as zero section. This group acts on X by translation (on each fiber), hence it leaves the two form invariant. We assume that the singular fibers of the fibration are all of type I_m, $m \in \mathbb{N}$. Let F_j be a fiber of type I_{m_j}, we denote by $C_0^{(j)}$ the irreducible components of the fibers meeting the zero section. After choosing an orientation, we denote the other irreducible components of the fibers by $C_1^{(j)}, \ldots, C_{m_j-1}^{(j)}$. In the sequel we always consider m_j a prime number, and the notation $C_i^{(j)}$ means $i \in \mathbb{Z}/m_j\mathbb{Z}$. For each section r we define the number $k := k_j(r)$ by

$$r \cdot C_j^{(k)} = 1 \text{ and } r \cdot C_j^{(i)} = 0 \text{ if } i = 0, \ldots, m_j - 1 \ \ i \neq k.$$

If the section r is a torsion section and h is the number of reducible fibers of type I_{m_j}, then by [Mi2, Proposition 3.1] we have

$$(4) \qquad \sum_{j=1}^{h} k_j(r) \left(1 - \frac{k_j(r)}{m_j}\right) = 4.$$

Moreover we recall the Shioda-Tate formula (cf.[Shio, Corollary 5.3] or [Mi1, p.70])

$$
(5) \qquad \text{rank}(NS(X)) = 2 + \sum_{j=1}^{h}(m_j - 1) + \text{rank}(MW_f).
$$

The $\text{rank}(MW_f)$ is the number of generators of the free part. If there are no sections of infinite order then $\text{rank}(MW_f) = 0$. Assume that X has h fibers of type I_m, $m \in \mathbb{N}$, $m > 1$, and the remaining singular fibers are of type I_1, which are rational curves with one node. Let $U \oplus (A_{m-1})^h$ denote the lattice generated by the zero section, the generic fiber and by the components of the reducible fibers not meeting s. If there are no sections of infinite order then it has finite index in $NS(X)$ equal to n, the order of the torsion part of the group MW_f. Using this remark we find that

$$
(6) \qquad |\det(NS(X))| = \frac{\det(A_{m-1})^h}{n^2} = \frac{m^h}{n^2}.
$$

3.1. **Elliptic fibrations with a symplectic automorphism.** Now we describe three particular elliptic fibrations which admit a symplectic automorphism σ_3, σ_5 or σ_7. Assume that we have a section of prime order $p = 3, 5, 7$. By [Shim, No. 560, 2346, 3256] there exist elliptic fibrations with one of the following configurations of components of singular fibers I_p not meeting s such that all the singular fibers of the fibrations are semistable (i.e. they are all of type I_n for a certain $n \in \mathbb{N}$) and the order of the torsion subgroup of the Mordell-Weil group $o(MW_f) = p$:

$$
(7) \qquad \begin{array}{lll}
p = 3 : & 6A_2 & o(MW_f) = 3, \\
p = 5 : & 4A_4 & o(MW_f) = 5, \\
p = 7 : & 3A_6 & o(MW_f) = 7.
\end{array}
$$

We can assume that the remaining singular fibers are of type I_1. Since the sum of the Euler characteristic of the fibers must add up to 24, these are six, four, resp. three fibers. Observe that each section of finite order induces a symplectic automorphism of the same order which corresponds to a translation by the section on each fiber, we denote it by σ_p. The nodes of the I_1 fibers are then the fixed points of these automorphisms, whence σ_p permutes the p components of the I_p fibers. For these fibrations we have rank $NS(X) = 14$, 18, 20 and dimensions of the moduli spaces six, two and zero, which is one less then the maximal possible dimension of the moduli space we have given in the Proposition 1.1.

3.1.1. *Weierstrass forms.* We compute the Weierstrass form for the elliptic fibration described in (7). When X is a K3 surface then this form is

$$
(8) \qquad y^2 = x^3 + A(t)x + B(t), \ t \in \mathbb{P}^1
$$

or in homogeneous coordinates

$$
(9) \qquad x_3 x_2^2 = x_1^3 + A(t)x_1 x_3^2 + B(t)x_3^3
$$

where $A(t)$ and $B(t)$ are polynomials of degrees eight and twelve respectively, $x_3 = 0$ is the line at infinity and also the tangent to the inflectional point $(0 : 1 : 0)$.
Fibration with a section of order 3. In this case the point of order three must be an inflectional point (cf. [C, Ex. 5, p.38]), we want to determine $A(t)$ and $B(t)$ in the

equation (8). We start by imposing to a general line $y = l(t)x + m(t)$ to be an inflectional tangent so the equation of the elliptic fibration is

$$y^2 = x^3 + A(t)x + B(t), \ t \in \mathbb{P}^1, \text{ with } \quad A(t) = \frac{2l(t)m(t) + l(t)^4}{3}, \ B(t) = \frac{m(t)^2 - l(t)^6}{3^3}.$$

Since $A(t)$ and $B(t)$ are of degrees eight and twelve, we have $\deg l(t) = 2$ and $\deg m(t) = 6$. The section of order three is

$$t \mapsto \left(\frac{l(t)^2}{3}, \frac{l(t)^3}{3} + m(t) \right).$$

The discriminant $\Delta = 4A^3 + 27B^2$ of the fibration is

$$\Delta = \frac{(5l(t)^3 + 27m(t))(l(t)^3 + 3m(t))^3}{27}$$

hence in general it vanishes to the order three on six values of t and to the order one on other six values. Since A and B in general do not vanish on these values, this equation parametrizes an elliptic fibration with six fibers I_3 (so we have six curves A_2 not meeting the zero section) and six fibers I_1 (cf. [Mi1, Table IV.3.1 pag.41]).

Fibration with a section of order 5. In the same way we can compute the Weierstrass form of the elliptic fibration described in (7) with a section of order five.

In [BM] a geometrical condition for the existence of a point of order five on an elliptic curve is given. For fixed t let the cubic curve be in the form (9) then take two arbitrary distinct lines through O which meet the cubic in two other distinct points each. Call 1, 4 the points on the first line and 2, 3 the points on the second line, then 1 (or any of the other point) has order five if:

-the tangent through 1 meets the cubic in 3,
-the tangent through 4 meets the cubic in 2,
-the tangent through 3 meets the cubic in 4,
-the tangent through 2 meets the cubic in 1.

These conditions give the Weierstrass form:

$$y^2 = x^3 + A(t)x + B(t), \ t \in \mathbb{P}^1, \text{ with}$$

$$A(t) = \frac{(-b(t)^4 + b(t)^2 a(t)^2 - a(t)^4 - 3a(t)b(t)^3 + 3a(t)^3 b(t))}{3},$$

$$B(t) = \frac{(b(t)^2 + a(t)^2)(19b(t)^4 - 34b(t)^2 a(t)^2 + 19a(t)^4 + 18a(t)b(t)^3 - 18a(t)^3 b(t))}{108}$$

where $\deg a(t) = 2$, $\deg b(t) = 2$. The section of order five is

$$t \mapsto ((2b(t)^2 - a(t)^2)2 : 3(a(t) + b(t))(a(t) - b(t))^2 : 6)$$

and the discriminant is

$$\Delta = \frac{1}{16}(b(t)^2 - a(t)^2)^5(11(b(t)^2 - a(t)^2) + 4a(t)b(t)).$$

By a careful analysis of the zeros of the discriminant we can see that the fibration has four fibers I_5 and four fibers I_1 (cf. [Mi1, Table IV.3.1 pag.41]).

Fibration with a section of order 7. To find the Weierstrass form we use also in this case the results of [BM]. We explain briefly the idea to find a set of points of order seven on an elliptic curve. One takes points $0, 3, 4$ and $1, 2, 4$ on two lines in the plane. Then the intersections of a line through 3 different from the lines $\{3, 2\}$, $\{3, 4\}$, $\{3, 1\}$ with the lines $\{1, 0\}$ and $\{2, 0\}$ give two new points -1 and -2. By using the conditions that the tangent

through 1 goes through -2 and the tangent through 2 goes through 3 one can determine a cubic having a point of order seven which is e.g. 1. By using these conditions one can find the equation, but since the computations are quite involved, we recall the Weierstrass form given in [T, p.195]

$$y^2 + (1 + t - t^2)xy + (t^2 - t^3)y = x^3 + (t^2 - t^3)x^2.$$

By a direct check one sees that the point of order seven is $(0(t), 0(t))$. This elliptic fibration has three fibers I_7 and three fibers I_1.

4. ELLIPTIC K3 SURFACES WITH AN AUTOMORPHISM OF PRIME ORDER

In this section we prove the main theorem:

Theorem 4.1. *For any K3 surface X with a symplectic automorphism σ_p of order $p = 2, 3, 5, 7$ the action on $H^2(X, \mathbb{Z})$ decomposes in the following way:*

$\mathbf{p = 2}:$ $H^2(X, \mathbb{Z})^{\sigma_2^*} = E_8(-2) \oplus U \oplus U \oplus U$, $(H^2(X, \mathbb{Z})^{\sigma_2^*})^\perp = E_8(-2)$.

$\mathbf{p = 3}:$ $H^2(X, \mathbb{Z})^{\sigma_3^*} = U \oplus U(3) \oplus U(3) \oplus A_2 \oplus A_2$

$$(H^2(X, \mathbb{Z})^{\sigma_3^*})^\perp = \left\{ (x_1, \ldots, x_6) \in (\mathbb{Z}[\omega_3])^{\oplus 6} : \begin{array}{l} x_i \equiv x_j \mod (1 - \omega_3), \\[2mm] \sum_{i=1}^6 x_i \equiv 0 \mod (1 - \omega_3)^2 \end{array} \right\} = K_{12}(-2)$$

with hermitian form $h(\alpha, \beta) = \sum_{i=1}^6 (\alpha_i \overline{\beta_i})$.

$\mathbf{p = 5}:$ $H^2(X, \mathbb{Z})^{\sigma_5^*} = U \oplus U(5) \oplus U(5)$

$$(H^2(X, \mathbb{Z})^{\sigma_5^*})^\perp = \left\{ (x_1, \ldots, x_4) \in (\mathbb{Z}[\omega_5])^{\oplus 4} : \begin{array}{l} x_1 \equiv x_2 \equiv 2x_3 \equiv 2x_4 \mod (1 - \omega_5), \\[2mm] (3 - \omega_5)(x_1 + x_2) + x_3 + x_4 \equiv 0 \mod (1 - \omega_5)^2 \end{array} \right\}$$

with hermitian form $h(\alpha, \beta) = \sum_{i=1}^2 \alpha_i \overline{\beta_i} + \sum_{j=3}^4 f\alpha_j \overline{f\beta_j}$ where $f = 1 - (\omega_5^2 + \omega_5^3)$.

$\mathbf{p = 7}:$ $H^2(X, \mathbb{Z})^{\sigma_7^*} = U(7) \oplus \begin{pmatrix} 4 & 1 \\ 1 & 2 \end{pmatrix}$

$$(H^2(X, \mathbb{Z})^{\sigma_7^*})^\perp = \left\{ (x_1, x_2, x_3) \in (\mathbb{Z}[\omega_7])^{\oplus 3} : \begin{array}{l} x_1 \equiv x_2 \equiv 6x_3 \mod (1 - \omega_7), \\[2mm] (1 + 5\omega_7)x_1 + 3x_2 + 2x_3 \equiv 0 \mod (1 - \omega_7)^2 \end{array} \right\}$$

with hermitian form $h(\alpha, \beta) = \alpha_1 \overline{\beta_1} + f_1 \alpha_2 \overline{f_1 \beta_2} + f_2 \alpha_3 \overline{f_2 \beta_3}$
where $f_1 = 3 + 2(\omega_7 + \omega_7^6) + (\omega_7^2 + \omega_7^5)$ and $f_2 = 2 + (\omega_7 + \omega_7^6)$.

In the case $p = 3$, $K_{12}(-2)$ denotes the Coxeter-Todd lattice with the bilinear form multiplied by -2.

This theorem gives a complete description of the invariant sublattice $H^2(X, \mathbb{Z})^{\sigma_p^*}$ and its orthogonal complement in $H^2(X, \mathbb{Z})$ for the symplectic automorphisms σ_p of all possible prime order $p = 2, 3, 5, 7$ acting on a K3 surface. The results about the order two automorphism is proven by Morrison in [Mo, Theorem 5.7].

We describe the lattices of the theorem and their hermitian forms in the sections from 4.1 to 4.7. The proof is the following: we identify the action of σ_p^* on $H^2(X, \mathbb{Z})$ in the case of X an elliptic K3 surface, this is done in several propositions in these sections, then we apply [Ni1, Theorem 4.7] which assure the uniqueness of this action.

4.1. **A section of order three.** Let X be a K3 surface with an elliptic fibration which admits a section of order three described in (7) of section 3. We recall that X has six reducible fibres of type I_3 and six singular irreducible fibres of type I_1. In the preceding section we have seen that the rank of the Néron-Severi group is 14. We determine now $NS(X)$ and T_X.

Let t_1 denote the section of order three and $t_2 = t_1 + t_1$. Let σ_3 be the automorphism of X which corresponds to the translation by t_1. It leaves each fiber invariant and $\sigma_3^*(s) = t_1$, $\sigma_3^*(t_1) = t_2$, $\sigma_3^*(t_2) = s$. Denoted by $C_0^{(i)}, C_1^{(i)}, C_2^{(i)}$ the components of the $i - th$ reducible fiber $(i = 1, \ldots, 6)$, we can assume that $C_1^{(i)} \cdot t_1 = C_2^{(i)} \cdot t_2 = C_0^{(i)} \cdot s = 1$.

Proposition 4.1. *A $\mathbb{Z}$-basis for the lattice $NS(X)$ is given by*

$$s, t_1, t_2, F, C_1^{(1)}, C_2^{(1)}, C_1^{(2)}, C_2^{(2)}, C_1^{(3)}, C_2^{(3)}, C_1^{(4)}, C_2^{(4)}, C_1^{(5)}, C_2^{(5)}.$$

Let $U \oplus A_2^6$ be the lattice generated by the section, the fiber and the irreducible components of the six fibers I_3 which do not intersect the zero section s. It has index three in the Néron-Severi group of X, $NS(X)$. The lattice $NS(X)$ has discriminant -3^4 and its discriminant form is

$$\mathbb{Z}_3(\frac{2}{3}) \oplus \mathbb{Z}_3(\frac{2}{3}) \oplus \mathbb{Z}_3(\frac{2}{3}) \oplus \mathbb{Z}_3(-\frac{2}{3}).$$

The transcendental lattice T_X is

$$T_X = U \oplus U(3) \oplus A_2 \oplus A_2$$

and has a unique primitive embedding in the lattice Λ_{K3}.

Proof. It is clear that a $\mathbb{Q}$-basis for $NS(X)$ is given by $s, F, C_1^{(i)}, C_2^{(i)}$, $i = 1, \ldots, 6$. This basis generates the lattice $U \oplus A_2^6$. It has discriminant $d(U \oplus A_2^6) = -3^6$. We denote by

$$c_i = 2C_1^{(i)} + C_2^{(i)}, \quad C = \sum c_i,$$
$$d_i = C_1^{(i)} + 2C_2^{(i)}, \quad D = \sum d_i.$$

Since we know that $t_1 \in NS(X)$ we can write

$$t_1 = \alpha s + \beta F + \sum \gamma_i C_1^{(i)} + \sum \delta_i C_2^{(i)}, \quad \alpha, \beta, \gamma_i, \delta_i \in \mathbb{Q}.$$

Then by using the fact that $t_1 \cdot s = t_1 \cdot C_2^{(i)} = 0$ and $t_1 \cdot C_1^{(i)} = t_1 \cdot F = 1$ one obtains that $\alpha = 1$, $\beta = 2$ and $\gamma_1 = -2/3$, $\delta_1 = -1/3$ hence $\frac{1}{3}C \in NS(X)$. A similar computation with t_2 shows that $\frac{1}{3}D \in NS(X)$. So one obtains that

(10)
$$t_1 = s + 2F - \tfrac{1}{3}C \in NS(X),$$
$$t_2 = s + 2F - \tfrac{1}{3}D \in NS(X).$$

and so

$$3(t_2 - t_1) = \sum_{i=1}^{6} (C_1^{(i)} - C_2^{(i)}) = C - D.$$

We consider now the $\mathbb{Q}$-basis for the Néron-Severi group

$$s, t_1, t_2, F, C_1^{(1)}, C_2^{(1)}, C_1^{(2)}, C_2^{(2)}, C_1^{(3)}, C_2^{(3)}, C_1^{(4)}, C_2^{(4)}, C_1^{(5)}, C_2^{(5)}.$$

By computing the matrix of the intersection form respect to this basis one finds that the determinant is -3^4. By the Shioda-Tate formula we have $|\det(NS(X))| = 3^4$. Hence this is a $\mathbb{Z}$-basis for the Néron-Severi group. We add to the classes which generate $U \oplus A_2^6$ the

classes t_1 and $\sigma_3^*(t_1) = t_2$ given in the formula (10). Since $d(U \oplus A_2^6) = 3^6$ and $d(NS(X)) = -3^4$ the index of $U \oplus A_2^6$ in $NS(X)$ is 3. Observe that this is also a consequence of a general result given at the end of section 3.

The classes

$$v_i = \frac{C_1^{(i)} - C_2^{(i)} - (C_1^{(5)} - C_2^{(5)})}{3}, \quad i = 1, \ldots, 4$$

generate the discriminant group, which is $NS(X)^\vee/NS(X) \cong (\mathbb{Z}/3\mathbb{Z})^{\oplus 4}$.

These classes are not orthogonal to each other with respect to the bilinear form, so we take

$$w_1 = v_1 - v_2, \ w_2 = v_3 - v_4, \ w_3 = v_1 + v_2 + v_3 + v_4, \ w_4 = v_1 + v_2 - (v_3 + v_4)$$

which form an orthogonal basis with respect to the bilinear form with values in $\mathbb{Q}/\mathbb{Z}$. And it is easy to compute that $w_1^2 = w_2^2 = w_3^2 = 2/3$, $w_4^2 = -2/3$. The discriminant form of the lattice $NS(X)$ is then

$$(11) \qquad\qquad \mathbb{Z}_3\left(\frac{2}{3}\right) \oplus \mathbb{Z}_3\left(\frac{2}{3}\right) \oplus \mathbb{Z}_3\left(\frac{2}{3}\right) \oplus \mathbb{Z}_3\left(-\frac{2}{3}\right).$$

The transcendental lattice T_X orthogonal to $NS(X)$ has rank eight. Since $NS(X)$ has signature $(1, 13)$, the transcendental lattice has signature $(2, 6)$. The discriminant form of the transcendental lattice is the opposite of the discriminant form of the Néron-Severi lattice. So the transcendental lattice has signature $(2, 6)$, discriminant 3^4, discriminant group $T_X^\vee/T_X \cong (\mathbb{Z}/3\mathbb{Z})^{\oplus 4}$ and discriminant form $\mathbb{Z}_3(-\frac{2}{3}) \oplus \mathbb{Z}_3(-\frac{2}{3}) \oplus \mathbb{Z}(-\frac{2}{3}) \oplus \mathbb{Z}_3(\frac{2}{3})$. By [Ni2, Cor. 1.13.5] we have $T = U \oplus T'$ where T' has rank six, signature $(1, 5)$ and T' has discriminant form as before. These data identify T' uniquely ([Ni2, Corollary 1.13.3]). Hence it is isomorphic to $U(3) \oplus A_2 \oplus A_2$ with generators for the discriminant form

$$(e - f)/3, \ (e + f)/3, \ (A - B)/3, \ (A' - B')/3,$$

where e, f, A, B, A', B' are the usual bases of the lattices.

The transcendental lattice

$$T_X = U \oplus U(3) \oplus A_2 \oplus A_2$$

has a unique embedding in the lattice Λ_{K3} by [Ni2, Theorem 1.14.4] or [Mo, Corollary 2.10]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

4.1.1. *The invariant lattice and its orthogonal complement.*

Proposition 4.2. *The invariant sublattice of the Néron-Severi group is isometric to $U(3)$ and it is generated by the classes F and $s + t_1 + t_2$.*
The invariant sublattice $H^2(X, \mathbb{Z})^{\sigma_3^*}$ is isometric to $U \oplus U(3) \oplus U(3) \oplus A_2 \oplus A_2$.
Its orthogonal complement $\Omega_3 := (H^2(X, \mathbb{Z})^{\sigma_3^*})^\perp$ is the negative definite twelve dimensional

lattice $\{\mathbb{Z}^{12}, M\}$ where M is the bilinear form

$$\begin{pmatrix}
-4 & 2 & -3 & -2 & 0 & -2 & 0 & -2 & 0 & -2 & 0 & -2 \\
2 & -4 & 3 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
-3 & 3 & -18 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & -9 \\
-2 & 1 & 0 & -6 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -3 & -4 & 3 & 2 & 0 & 0 & 0 & 0 & 0 \\
-2 & 1 & 0 & 0 & 3 & -6 & -3 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & -3 & -4 & 3 & 2 & 0 & 0 & 0 \\
-2 & 1 & 0 & 0 & 0 & 0 & 3 & -6 & -3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2 & -3 & -4 & 3 & 2 & 0 \\
-2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & -6 & -3 & 0 \\
0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 2 & -3 & -4 & 3 \\
-2 & 1 & -9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & -6
\end{pmatrix}$$

and it is equal to the lattice $(NS(X)^{\sigma_3^*})^{\perp}$.
The lattice Ω_3 admits a unique primitive embedding in the lattice Λ_{K3}.
The discriminant of Ω_3 is 3^6 and its discriminant form is $(\mathbb{Z}_3(\frac{2}{3}))^{\oplus 6}$.
The isometry σ_3^* acts on the discriminant group $\Omega_3^{\vee}/\Omega_3$ as the identity.

Proof. It is clear that the isometry σ_3^* fixes the classes F and $s + t_1 + t_2$. These generate a lattice $U(3)$ (with basis F and $F + s + t_1 + t_2$).
The invariant sublattice $H^2(X, \mathbb{Z})^{\sigma_3^*}$ contains T_X and the invariant sublattice of the Néron-Severi group. So $(H^2(X, \mathbb{Z})^{\sigma_3^*})^{\perp} = (NS(X)^{\sigma_3^*})^{\perp}$, this lattice has signature $(0, 12)$ and by [Ni1, p. 133] the discriminant group is $(\mathbb{Z}/3\mathbb{Z})^{\oplus 6}$. Hence by [Ni2, Theorem 1.14.4] there is a unique primitive embedding of $(H^2(X, \mathbb{Z})^{\sigma_3^*})^{\perp}$ in the K3-lattice. By using the orthogonality conditions one finds the following basis of $\Omega_3 = (NS(X)^{\sigma_3^*})^{\perp}$:

$$b_1 = t_2 - t_1, \quad b_2 = s - t_2, \quad b_3 = F - 3C_2^{(5)}, \quad b_{2(i+1)} = C_1^{(i)} - C_2^{(i)}, \quad i = 1, \ldots, 5$$
$$b_{2j+3} = C_1^{(j)} - C_1^{(j+1)}, \quad j = 1, \ldots, 4.$$

An easy computation shows that the Gram matrix of this basis is exactly the matrix M which indeed has determinant 3^6.
Since $H^2(X, \mathbb{Z})^{\sigma_3^*} \supseteq T_X \oplus NS(X)^{\sigma_3^*} = U \oplus U(3) \oplus U(3) \oplus A_2 \oplus A_2$ and these lattices have the same rank, to prove that the inclusion is an equality we compare their discriminants. The lattice $(H^2(X, \mathbb{Z})^{\sigma_3^*})^{\perp}$ has determinant 3^6. So the lattice $(H^2(X, \mathbb{Z})^{\sigma_3^*})$ has determinant -3^6 (because these are primitive sublattices of $H^2(X, \mathbb{Z})$). The lattice $U \oplus U(3) \oplus U(3) \oplus A_2 \oplus A_2$ has determinant exactly -3^6, so

$$H^2(X, \mathbb{Z})^{\sigma_3^*} = U \oplus U(3) \oplus U(3) \oplus A_2 \oplus A_2.$$

Since $NS(X)^{\vee}/NS(X) \subset \Omega_3^{\vee}/\Omega_3$ the generators of the discriminant form of the lattice Ω_3 are classes $w_1, \ldots, w_6$ with $w_1, \ldots, w_4$ the classes which generate the discriminant form of $NS(X)$ (cf. the proof of the Proposition 4.1) and

$$w_5 = \frac{1}{3}(b_1 + 2b_2) = \frac{1}{3}(2s - t_1 - t_2) \quad w_6 = \frac{1}{3}(b_1 + 2b_2 - 2b_3) = \frac{1}{3}(2s - t_1 - t_2 - 2F + 6C_2^{(5)}).$$

These six classes are orthogonal, with respect to the bilinear form taking values in $\mathbb{Q}/\mathbb{Z}$, and generate the discriminant form. Their squares are $w_1^2 = w_2^2 = w_3^2 = w_5^2 \equiv \frac{2}{3}$ mod $2\mathbb{Z}$, $w_4^2 = w_6^2 \equiv -\frac{2}{3}$ mod $2\mathbb{Z}$. By replacing w_4, w_6 by $w_4 - w_6, w_4 + w_6$ we obtain the discriminant form $(\mathbb{Z}_3(\frac{2}{3}))^{\oplus 6}$.
By computing the image of w_i, $i = 1, \ldots, 6$ under σ_3^* one finds that $\sigma_3^*(w_i) - w_i \in \Omega_3$. For

example: $\sigma_3^*(w_5) - w_5 = \frac{1}{3}(2t_1 - t_2 - s) - \frac{1}{3}(2s - t_1 - t_2) = t_1 - s$ which is an element of Ω_3 (in fact it is orthogonal to F and to $s + t_1 + t_2$). Hence the action of σ_3^* is trivial on $\Omega_3^\vee / \Omega_3$ as claimed. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

In the next two subsections we apply the results of section 2 about the $\mathbb{Z}[\omega]$-lattices to describe the lattice $\{\Omega_3, M\}$ and to prove that Ω_3 is isomorphic to the lattice $K_{12}(-2)$, where K_{12} is the Coxeter-Todd lattice (cf. e.g. [CT], [CS] for a description of this lattice).

4.2. **The lattice Ω_3.** Let ω_3 be a primitive third root of the unity. In this section we prove the following result (we use the same notations of section 2):

Theorem 4.2. *The lattice Ω_3 is isometric to the $\mathbb{Z}$-lattice associated to the $\mathbb{Z}[\omega_3]$-lattice $\{L_3,\ h_{L_3}\}$ where*

$$L_3 = \left\{ (x_1, \dots, x_6) \in (\mathbb{Z}[\omega_3])^{\oplus 6}\ :\ \begin{array}{c} x_i \equiv x_j \mod (1 - \omega_3), \\[2mm] \sum_{i=1}^{6} x_i \equiv 0 \mod (1 - \omega_3)^2 \end{array} \right\}$$

and h_{L_3} is the restriction of the standard hermitian form on $\mathbb{Z}[\omega_3]^{\oplus 6}$.

Proof. Let $F = F_3^6$ be the $\mathbb{Z}$-sublattice of $NS(X)$ generated by

$$C_i^{(j)} - C_{i+1}^{(j)}, \quad i = 0, 1, 2, \quad j = 1, \dots, 6$$

with bilinear form induced by the intersection form on $NS(X)$.

Let $G = G_3^6$ denote the $\mathbb{Z}[\omega_3]$-lattice $(1 - \omega_3)^2 \mathbb{Z}[\omega_3]^{\oplus 6}$ with the standard hermitian form. This is a sublattice of $\mathbb{Z}[\omega_3]^{\oplus 6}$. Applying to each component of G the Lemma 2.1 we know that $\{G_{\mathbb{Z}}, b_G\}$ is a $\mathbb{Z}$-lattice isometric to the lattice F. The explicit isometry is given by

$$\begin{aligned} C_i^{(1)} - C_{i+1}^{(1)} &\mapsto (1 - \omega_3)^2 (\omega_3^{i-1}, 0, 0, 0, 0, 0) \\ C_i^{(2)} - C_{i+1}^{(2)} &\mapsto (1 - \omega_3)^2 (0, \omega_3^{i-1}, 0, 0, 0, 0) \\ &\vdots \\ C_i^{(6)} - C_{i+1}^{(6)} &\mapsto (1 - \omega_3)^2 (0, 0, 0, 0, 0, \omega_3^{i-1}). \end{aligned}$$

The multiplication by ω_3 of an element $(1 - \omega_3)^2 e_j$ (where e_j is the canonical basis) corresponds to a translation by t_1 on a singular fiber, which sends the curve $C_i^{(j)}$ to the curve $C_{i+1}^{(j)}$. Hence we have a commutative diagram:

$$\begin{array}{ccc} F & \longrightarrow & G \\ \sigma_3^* \downarrow & & \downarrow \ \cdot \omega_3 \\ F & \longrightarrow & G. \end{array}$$

The elements $C_i^{(j)} - C_k^{(j)}$, $i, k = 0, 1, 2$, $j = 1, \dots, 6$ are all contained in the lattice $\Omega_3 = (NS(X)^{\sigma_3^*})^\perp$, but they do not generate this lattice. A set of generators for Ω_3 is

$$s - t_1, \quad t_1 - t_2, \quad C_i^{(j)} - C_h^{(k)} \quad i, h = 0, 1, 2, \ j, k = 1, \dots, 6.$$

From the formula (10) we obtain that

$$s - t_1 = \sum_{j=1}^{6} [\frac{1}{3}(C_1^{(j)} - C_2^{(j)}) + \frac{1}{3}\sigma_3^*(C_1^{(j)} - C_2^{(j)})].$$

After the identification of F with $G_{\mathbb{Z}}$ we have

$$s - t_1 = (1 - \omega_3)^2(\frac{1}{3}(1 + \omega_3))(1, 1, 1, 1, 1, 1) = (1, 1, 1, 1, 1, 1).$$

The divisor $t_1 - t_2$, which is the image of $s - t_1$ under the action of σ_3^*, corresponds to the vector $(\omega_3, \omega_3, \omega_3, \omega_3, \omega_3, \omega_3)$. Similarly one can see that the element $C_1^{(1)} - C_1^{(2)}$ corresponds to the vector $(1 - \omega_3)(1, -1, 0, 0, 0, 0)$ and more in general $C_i^{(j)} - C_i^{(k)}$ with $j \neq k$ corresponds to the vector $(1 - \omega_3)(\omega_3^{i-1}e_j - \omega_3^{i-1}e_k)$ where e_i is the standard basis. The lattice L_3 generated by the vectors of $G_{\mathbb{Z}}$ and by

$$\omega_3^i(1, 1, 1, 1, 1, 1) \quad (1 - \omega_3)\omega_3^{i-1}(-e_j + \omega_3 e_k) \quad i = 0, 1, 2, \quad j, k = 1, \ldots 6,$$

is thus isometric to Ω_3.
In conclusion a basis for L_3 is

$$
\begin{aligned}
l_1 &= -\omega_3(1, 1, 1, 1, 1, 1) & l_2 &= -\omega_3^2(1, 1, 1, 1, 1, 1) = (1 + \omega_3)(1, 1, 1, 1, 1, 1) \\
l_3 &= (1 - \omega_3)^2(0, 0, 0, 0, 1 - \omega_3, 0) & l_4 &= (1 - \omega_3)^2(1, 0, 0, 0, 0, 0) \\
l_5 &= (1 - \omega_3)(1, -1, 0, 0, 0, 0) & l_6 &= (1 - \omega_3)^2(0, 1, 0, 0, 0, 0) \\
l_7 &= (1 - \omega_3)(0, 1, -1, 0, 0, 0) & l_8 &= (1 - \omega_3)^2(0, 0, 1, 0, 0, 0) \\
l_9 &= (1 - \omega_3)(0, 0, 1, -1, 0, 0) & l_{10} &= (1 - \omega_3)^2(0, 0, 0, 1, 0, 0) \\
l_{11} &= (1 - \omega_3)(0, 0, 0, 1, -1, 0) & l_{12} &= (1 - \omega_3)^2(0, 0, 0, 0, 1, 0).
\end{aligned}
$$

The identification between Ω_3 and L_3 is given by the map $b_i \mapsto l_i$.
After this identification the intersection form on Ω_3 is exactly the form $b_{|L_3}$ on L_3. The basis l_i of L_3 satisfies the condition given in the theorem, and so

$$L_3 \subseteq \quad \{(x_1, \ldots, x_6) \in (\mathbb{Z}[\omega_3])^{\oplus 6} \; : \; x_i \equiv x_j \quad \mod (1 - \omega_3),$$
$$\textstyle\sum_{i=1}^6 x_i \equiv 0 \quad \mod (1 - \omega_3)^2\}.$$

Since the vectors $(1 - \omega_3)^2 e_j$, $(1 - \omega_3)(e_i - e_j)$ and $(1, 1, 1, 1, 1, 1)$ generate the $\mathbb{Z}[\omega_3]$-lattice $\{(x_1, \ldots, x_6) \in (\mathbb{Z}[\omega_3])^{\oplus 6} \; : \; x_i \equiv x_j \quad \mod (1 - \omega_3), \quad \sum_{i=1}^6 x_i \equiv 0 \quad \mod (1 - \omega_3)^2\}$ and since they are all vectors contained in L_3, the equality holds. $\square$

4.3. The Coxeter-Todd lattice K_{12}.

Theorem 4.3. *The lattice Ω_3 is isometric to the lattice $K_{12}(-2)$.*

Proof. The lattice K_{12} is described by Coxeter and Todd in [CT] and by Conway and Sloane in [CS]. The lattice K_{12} is the twelve dimensional $\mathbb{Z}$-module associated to a six dimensional $\mathbb{Z}[\omega_3]$-lattice $\Lambda_6^{\omega_3}$.
The $\mathbb{Z}[\omega_3]$-lattice $\Lambda_6^{\omega_3}$ is described in [CS] in four different ways. We recall one of them denoted by $\Lambda^{(3)}$ in [CS, Definition 2.3], which is convenient for us. Let $\theta = \omega_3 - \bar{\omega}_3$, then $\Lambda_6^{\omega_3}$ is the $\mathbb{Z}[\omega_3]$-lattice

$$\Lambda_6^{\omega_3} = \{(x_1, \ldots, x_6) : \; x_i \in \mathbb{Z}[\omega_3], x_i \equiv x_j \quad \mod \theta, \; \sum_{i=1}^6 x_i \equiv 0 \quad \mod 3\}$$

with hermitian form $\frac{1}{3}{}^t x \bar{y}$. We observe that $\theta = \omega_3(1 - \omega_3)$. The element ω_3 is a unit in $\mathbb{Z}[\omega_3]$ so the congruence modulo θ is the same as the congruence modulo $(1 - \omega_3)$. Observing that $-3 = \theta^2$ it is then clear that the $\mathbb{Z}[\omega_3]$-module $\Lambda_6^{\omega_3}$ is the $\mathbb{Z}[\omega_3]$-module L_3. The $\mathbb{Z}$-modules K_{12} and $L_{3,\mathbb{Z}}$ are isomorphic since they are the twelve dimensional

$\mathbb{Z}$-lattices associated to the same $\mathbb{Z}[\omega_3]$-lattice. The bilinear form on the $\mathbb{Z}$-module K_{12} is given by

$$b_{K_{12}}(x,y) = \frac{1}{3}x\bar{y} = \frac{1}{6}Tr(x\bar{y})$$

and the bilinear form on $L_{3,\mathbb{Z}}$ is given by

$$b_{|L_3}(x,y) = -\frac{1}{3}Tr(x\bar{y}).$$

So the $\mathbb{Z}$-lattice $\{L_{3,\mathbb{Z}}, b_{|L_3}\}$ is isometric to $K_{12}(-2)$. $\square$

Remark. 1) In [CT] Coxeter and Todd give an explicit basis of the $\mathbb{Z}$-lattice K_{12}. By a direct computation one can find the change of basis between the basis described in [CT] and the basis $\{b_i\}$ given in the proof of Proposition 4.2.
2) The lattice Ω_3 does not contain vectors of norm -2 (cf. [Ni1, Lemma 4.2]), but has 756 vectors of norm -4, 4032 of norm -6 and 20412 of norm -8. Since these properties define the lattice $K_{12}(-2)$, (cf. [CS, Theorem 1]), this is another way to prove the equality between Ω_3 and $K_{12}(-2)$.
3) The lattice $K_{12}(-2)$ is generated by vectors of norm -4, [PP, Section 3].

4.4. **Section of order five.** Let X be a K3 surface with an elliptic fibration which admits a section of order five as described in section 3. We recall that X has four reducible fibres of type I_5 and four singular irreducible fibres of type I_1. We have seen that the rank of the Néron-Severi group is 18. We determine now $NS(X)$ and T_X.
We label the fibers and their components as described in the section 3. Let t_1 denote the section of order five which meets the first singular fiber in $C_1^{(1)}$. By the formula (4) of section 3 up to permutation of the fibers only the following situations are possible:

$$t_1 \cdot C_1^{(1)} = t_1 \cdot C_1^{(2)} = t_1 \cdot C_2^{(3)} = t_1 \cdot C_2^{(4)} = 1 \text{ and } t_1 \cdot C_i^{(j)} = 0 \text{ otherwise;}$$
$$\text{or } t_1 \cdot C_1^{(1)} = t_1 \cdot C_4^{(2)} = t_1 \cdot C_2^{(3)} = t_1 \cdot C_3^{(4)} = 1 \text{ and } t_1 \cdot C_i^{(j)} = 0 \text{ otherwise.}$$

Observe that these two cases describe the same situation if we change the "orientation" on the last two fibers, so we assume to be in the first case. Let σ_5 be the automorphism of order five which leaves each fiber invariant and is the translation by t_1, so $\sigma_5^*(s) = t_1$, $\sigma_5^*(t_1) = t_2$, $\sigma_5^*(t_2) = t_3$, $\sigma_5^*(t_3) = t_4$, $\sigma_5^*(t_4) = s$.

Proposition 4.3. *A $\mathbb{Z}$-basis for the lattice $NS(X)$ is given by*

$$s, t_1, t_2, t_3, t_4, F, C_1^{(1)}, C_2^{(1)}, C_3^{(1)}, C_4^{(1)}, C_1^{(2)}, C_2^{(2)}, C_3^{(2)}, C_4^{(2)}, C_1^{(3)}, C_2^{(3)}, C_3^{(3)}, C_4^{(3)}.$$

Let $U \oplus A_4^4$ be the lattice generated by the section, the fiber and the irreducible components of the four fibers I_5 which do not intersect the zero section s. It has index five in the Néron-Severi group of X, $NS(X)$.
The lattice $NS(X)$ has discriminant -5^2 and its discriminant form is

$$\mathbb{Z}_5(\frac{2}{5}) \oplus \mathbb{Z}_5(-\frac{2}{5}).$$

The transcendental lattice is

$$T_X = U \oplus U(5)$$

and has a unique primitive embedding in the lattice Λ_{K3}.

Proof. The proof is similar to the proof of Proposition 4.1. So we sketch it briefly. The classes $s, F, C_i^{(j)}$, $i = 1, \ldots, 4$, $j = 1, \ldots, 4$, generate $U \oplus A_4^4$. By using the intersection form, or by the result of [Mi2, p. 299], we find

$$
(12) \qquad
\begin{aligned}
t_1 = {} & s + 2F - \tfrac{1}{5}\Big[\textstyle\sum_{i=1}^2 (4C_1^{(i)} + 3C_2^{(i)} + 2C_3^{(i)} + C_4^{(i)}) + \\
& + \sum_{j=3}^4 (3C_2^{(j)} + 6C_2^{(j)} + 4C_3^{(j)} + 2C_4^{(j)})\Big].
\end{aligned}
$$

A $\mathbb{Z}$-basis is $s, t_1, t_2, t_3, t_4, F, C_1^{(1)}, C_2^{(1)}, C_3^{(1)}, C_4^{(1)}, C_1^{(2)}, C_2^{(2)}, C_3^{(2)}, C_4^{(2)}, C_1^{(3)}, C_2^{(3)}, C_3^{(3)}, C_4^{(3)}$. Since $d(NS(X)) = -5^2$ and $d(U \oplus A_4^4) = -5^4$, the index of $U \oplus A_4^4$ in $NS(X)$ is five. Let w_1 and w_2 be

$$
\begin{aligned}
w_1 &= \tfrac{1}{5}(2C_1^{(1)} + 4C_2^{(1)} + C_3^{(1)} + 3C_4^{(1)} + 4C_1^{(3)} + 3C_2^{(3)} + 2C_3^{(3)} + C_4^{(3)}); \\
w_2 &= \tfrac{1}{5}(3C_1^{(2)} + C_2^{(2)} + 4C_3^{(2)} + 2C_4^{(2)} + C_1^{(3)} + 2C_2^{(3)} + 3C_3^{(3)} + 4C_4^{(3)}).
\end{aligned}
$$

The classes $v_1 = w_1 - w_2$, $v_2 = w_1 + w_2$ are orthogonal classes and generate the discriminant group of $NS(X)$, the discriminant form is

$$
\mathbb{Z}_5\!\left(\frac{2}{5}\right) \oplus \mathbb{Z}_5\!\left(-\frac{2}{5}\right).
$$

The transcendental lattice T_X has rank four, signature $(2, 2)$ and discriminant form $\mathbb{Z}_5(-\tfrac{2}{5}) \oplus \mathbb{Z}_5(\tfrac{2}{5})$. Since in this case T_X is uniquely determined by signature and discriminant form (cf. [Ni2, Corollary 1.13.3]) this is the lattice

$$
T_X = U \oplus U(5).
$$

The transcendental lattice has a unique embedding in the lattice Λ_{K3} by [Ni2, Theorem 1.14.4] or [Mo, Corollary 2.10]. $\qquad\square$

4.4.1. *The invariant lattice and its orthogonal complement.*

Proposition 4.4. *The invariant sublattice of the Néron-Severi lattice is isometric to the lattice $U(5)$ and it is generated by the classes F and $s + t_1 + t_2 + t_3 + t_4$.*
The invariant lattice $H^2(X, \mathbb{Z})^{\sigma_5^*}$ is isometric to $U \oplus U(5) \oplus U(5)$ and its orthogonal complement $\Omega_5 = (H^2(X, \mathbb{Z})^{\sigma_5^*})^{\perp}$ is the negative definite sixteen dimensional lattice $\{\mathbb{Z}^{16}, M\}$ where M is the bilinear form

$$
\begin{pmatrix}
-4 & 2 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 1 & -1 & 0 \\
2 & -4 & 2 & 0 & 5 & 2 & -1 & 0 & 0 & 2 & -1 & 0 & 1 & -1 & 1 & 1 \\
0 & 2 & -4 & 2 & -5 & -1 & 2 & -1 & 0 & -1 & 2 & -1 & 1 & -1 & 0 & -1 \\
0 & 0 & 2 & -4 & 0 & 0 & -1 & 2 & 0 & 0 & -1 & 2 & -1 & 1 & 1 & -1 \\
0 & 5 & -5 & 0 & -50 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & -15 \\
-1 & 2 & -1 & 0 & 0 & -6 & 4 & -1 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 2 & -1 & 0 & 4 & -6 & 4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 2 & 0 & -1 & 4 & -6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -3 & 1 & 0 & -4 & 3 & -1 & 0 & 2 & 0 & 0 & 0 \\
-1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 3 & -6 & 4 & -1 & -3 & 0 & 0 & 0 \\
0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & -1 & 4 & -6 & 4 & 1 & 0 & 0 & 0 \\
0 & 0 & -1 & 2 & 0 & 0 & 0 & 0 & 0 & -1 & 4 & -6 & 0 & 0 & 0 & 0 \\
-1 & 1 & 1 & -1 & 0 & 0 & 0 & 0 & 2 & -3 & 1 & 0 & -4 & 3 & -1 & 0 \\
1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & -6 & 4 & -1 \\
-1 & 1 & 0 & 1 & 5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 4 & -6 & 4 \\
0 & 1 & -1 & -1 & -15 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 4 & -6
\end{pmatrix}
$$

and it is equal to the lattice $(NS(X)^{\sigma_5^*})^{\perp}$.

The lattice Ω_5 *admits a unique primitive embedding in the lattice* Λ_{K3}.

The discriminant of Ω_5 *is* 5^4 *and its discriminant form is* $(\mathbb{Z}_5(\frac{2}{5}))^{\oplus 4}$.

The isometry σ_5^* *acts on the discriminant group* $\Omega_5^{\vee}/\Omega_5$ *as the identity.*

Proof. As in the case of an elliptic fibration with a section of order three, it is clear that σ_5^* fixes the classes F and $s + t_1 + t_2 + t_3 + t_4$. These classes generate the lattice $U(5)$, and so $H^2(X, \mathbb{Z})^{\sigma_5^*} \supseteq U(5) \oplus T_X = U(5) \oplus U(5) \oplus U$. Using Nikulin's result in [Ni1, p. 133] we find that the lattice $H^2(X, \mathbb{Z})^{\sigma_5^*}$ has determinant -5^4, which is exactly the determinant of $U(5) \oplus U(5) \oplus U$. Since these have the same rank, we conclude that $H^2(X, \mathbb{Z})^{\sigma_5^*} = U(5) \oplus U(5) \oplus U$.

The orthogonal complement $(H^2(X, \mathbb{Z})^{\sigma_5^*})^{\perp}$ is equal to $(NS(X)^{\sigma_5^*})^{\perp}$ as in Proposition 4.2. It has signature $(0, 16)$ and by [Ni1, p. 133] the discriminant group is $(\mathbb{Z}/5\mathbb{Z})^{\oplus 4}$. Hence by [Ni2, Theorem 1.14.4] there is a unique primitive embedding of $(H^2(X, \mathbb{Z})^{\sigma_5^*})^{\perp}$ in the K3-lattice. By using the orthogonality conditions one finds the following basis of $\Omega_5 = (NS(X)^{\sigma_5^*})^{\perp}$:

$$b_1 = s - t_1, \ b_2 = t_1 - t_2, \ b_3 = t_2 - t_3, \ b_4 = t_3 - t_4, \ b_5 = F - 5C_4^{(3)},$$
$$b_i = C_{i-5}^{(1)} - C_{i-4}^{(1)}, \ i = 6, 7, 8, \quad b_9 = C_1^{(1)} - C_1^{(2)},$$
$$b_i = C_{i-9}^{(2)} - C_{i-8}^{(2)}, \ i = 10, 11, 12, \quad b_{13} = C_1^{(2)} - C_1^{(3)},$$
$$b_i = C_{i-13}^{(3)} - C_{i-12}^{(3)}, \ i = 14, 15, 16.$$

The Gram matrix of this basis is exactly the matrix M.

The generators of the discriminant group of Ω_5 are the classes v_1, v_2 of the discriminant form of $NS(X)$ and the classes

$$v_3 = \tfrac{1}{5}(b_3 + 2b_1 + 3b_4 + 4b_2),$$
$$v_4 = \tfrac{1}{5}(b_3 + 2b_1 + 3b_4 + 4b_2 - b_5).$$

These have $v_3^2 = -2/5 \mod 2\mathbb{Z}$, $v_4^2 = 2/5 \mod 2\mathbb{Z}$. The generators v_1, $2v_2 - 4v_3 - v_4$, $2v_3$, v_4 are orthogonal to each other and have self-intersection $2/5$. $\square$

4.5. **The lattice** Ω_5. Let ω_5 be a primitive fifth root of the unity. In this section we prove the following result

Theorem 4.4. *The lattice* Ω_5 *is isometric to the* $\mathbb{Z}$-*lattice associated to the* $\mathbb{Z}[\omega_5]$-*lattice* $\{L_5, \ h_{L_5}\}$ *where*

$$L_5 = \left\{ (x_1, \ldots, x_4) \in (\mathbb{Z}[\omega_5])^{\oplus 4} : \begin{array}{l} x_1 \equiv x_2 \equiv 2x_3 \equiv 2x_4 \quad \mod (1 - \omega_5) \\[8pt] (3 - \omega_5)x_1 + (3 - \omega_5)x_2 + x_3 + x_4 \equiv 0 \quad \mod (1 - \omega_5)^2 \end{array} \right\}$$

with the hermitian form

$$(13) \qquad h_{L_5}(\alpha, \beta) = \sum_{i=1}^{2} \alpha_i \bar{\beta}_i + \sum_{j=3}^{4} f\alpha_j \overline{f\beta_j} = \sum_{i=1}^{2} \alpha_i \bar{\beta}_i + \tau \sum_{j=3}^{4} \alpha_j \overline{\beta}_j \ ,$$

where $\alpha, \beta \in L_5 \subset \mathbb{Z}[\omega_5]^{\oplus 4}$, $f = 1 - (\omega_5^2 + \omega_5^3)$ *and* $\tau = f\bar{f} = 2 - 3(\omega_5^2 + \omega_5^3)$.

Proof. The strategy of the proof is the same as in the case with an automorphism of order three, but the situation is more complicated because the section t_1 does not meet all the fibers I_5 in the same component. For this reason the hermitian form of the $\mathbb{Z}[\omega_5]$-lattice L_5 is not the standard hermitian form on all the components. It is possible to repeat the

construction used in the case of order three, but with the hermitian form (13). We explain now how we find this hermitian form.

Let $F := F_5^4$ be the lattice generated by the elements $C_i^{(j)} - C_{i+1}^{(j)}$, $i = 0, \ldots, 4$, $j = 1, \ldots, 4$. This is a sublattice of $(NS(X)^{\sigma_5*})^\perp$. A basis is

$$d_{1+i} = (\sigma_5^*)^i (C_1^{(1)} - C_2^{(1)}), \quad d_{5+i} = (\sigma_5^*)^i (C_1^{(2)} - C_2^{(2)}), \quad d_{9+i} = (\sigma_5^*)^i (C_1^{(3)} - C_2^{(3)}),$$
$$d_{13+i} = (\sigma_5^*)^i (C_1^{(4)} - C_2^{(4)}), \quad i = 0, \ldots, 3$$

and the bilinear form is thus the diagonal block matrix $Q = \operatorname{diag}(A, A, B, B)$

$$A = \begin{pmatrix} -6 & 4 & -1 & -1 \\ 4 & -6 & 4 & -1 \\ -1 & 4 & -6 & 4 \\ -1 & -1 & 4 & -6 \end{pmatrix}, \quad B = \begin{pmatrix} -6 & -1 & 4 & 4 \\ -1 & -6 & -1 & 4 \\ 4 & -1 & -6 & -1 \\ 4 & 4 & -1 & -6 \end{pmatrix}.$$

We want to identify the multiplication by ω_5 in the lattice G with the action of the isometry σ_5^* on the lattice F. We consider the $\mathbb{Z}[\omega_5]$-module $G = (1 - \omega)^2 \mathbb{Z}[\omega]^{\oplus 4}$. Now we consider the $\mathbb{Z}$-module $G_{\mathbb{Z}}$. The map

$$\phi: \quad \begin{aligned} (\sigma_5^*)^i (C_1^{(1)} - C_2^{(1)}) &\mapsto (1 - \omega_5)^2 \omega_5^i (1, 0, 0, 0) \\ (\sigma_5^*)^i (C_1^{(2)} - C_2^{(2)}) &\mapsto (1 - \omega_5)^2 \omega_5^i (0, 1, 0, 0) \\ (\sigma_5^*)^i (C_1^{(3)} - C_2^{(3)}) &\mapsto (1 - \omega_5)^2 \omega_5^i (0, 0, 1, 0) \\ (\sigma_5^*)^i (C_1^{(4)} - C_2^{(4)}) &\mapsto (1 - \omega_5)^2 \omega_5^i (0, 0, 0, 1) \end{aligned}$$

is an isomorphism between the $\mathbb{Z}$-modules $G_{\mathbb{Z}}$ and F.

Now we have to find a bilinear form b_G on G such that $\{G_{\mathbb{Z}}, b_G\}$ is isometric to $\{F, Q\}$. On the first and second fiber the action of σ_5^* is $\sigma_5^*(C_i^{(j)}) = C_{i+1}^{(j)}$, $j = 1, 2$, $i = 0, \ldots 4$, so $(\sigma_5^*)^i (C_1^{(j)} - C_2^{(j)}) = C_{i+1}^{(j)} - C_{i+2}^{(j)}$. Hence the map ϕ operates on the first two fibers in the following way:

$$\phi: \quad \begin{aligned} C_{i+1}^{(1)} - C_{i+2}^{(1)} &\mapsto (1 - \omega_5)^2 \omega_5^i (1, 0, 0, 0) \\ C_{i+1}^{(2)} - C_{i+2}^{(2)} &\mapsto (1 - \omega_5)^2 \omega_5^i (0, 1, 0, 0). \end{aligned}$$

This identification is exactly the identification described in Lemma 2.1, so on these generators of the lattices F and G we can choose exactly the form described in the lemma.

On the third and fourth fiber the action of σ_5^* is different (because σ_5^* is the translation by t_1 and it meets the first and second fiber in the component C_1 and the third and fourth fiber in the component C_2). In fact $(\sigma_5^*)^i (C_1^{(j)} - C_2^{(j)}) = C_{2i+1}^{(j)} - C_{2i+2}^{(j)}$, $j = 3, 4$, $i = 0, \ldots, 4$ and so

$$(14) \qquad \phi: \quad \begin{aligned} C_{2i+1}^{(3)} - C_{2i+2}^{(3)} &\mapsto (1 - \omega_5)^2 \omega_5^i (0, 0, 1, 0) \\ C_{2i+1}^{(4)} - C_{2i+2}^{(4)} &\mapsto (1 - \omega_5)^2 \omega_5^i (0, 0, 0, 1). \end{aligned}$$

A direct verification shows that the map ϕ defines an isometry between the module generated by $(\sigma_p^*)^i (C_1^{(j)} - C_2^{(j)})$, $i = 0, \ldots, 4$ and $(1 - \omega_5)^2 \mathbb{Z}[\omega_5]$, $(j = 3, 4)$ if one considers on $(1 - \omega_5)^2 \mathbb{Z}[\omega_5]$ the bilinear form associated to the hermitian form $h(\alpha, \beta) = \tau \alpha \overline{\beta}$ where $\tau = (2 - 3(\omega_5^2 + \omega_5^3))$. The real number τ is the square of $f = 1 - (\omega_5^2 + \omega_5^3)$, so the hermitian form above is also $h(\alpha, \beta) = \tau \alpha \overline{\beta} = f \alpha \overline{f \beta}$. So now we consider the $\mathbb{Z}[\omega_5]$-lattice $\mathbb{Z}[\omega_5]^{\oplus 4}$ with the hermitian form h given in (13) and G as a sublattice of $\{\mathbb{Z}[\omega_5]^{\oplus 4}, h\}$. We show that $L_5 = \Omega_5$. We have to add to the lattice F some classes to obtain the lattice Ω_5, and so we have to add some vectors to the lattice G to obtain the lattice L_5. It is sufficient to

add to F the classes $s - t_1$, $C_1^{(1)} - C_1^{(2)}$, $C_1^{(2)} - C_1^{(3)}$, $C_1^{(3)} - C_1^{(4)}$ and their images under σ_5^*. These classes correspond to the following vectors in $\mathbb{Z}[\omega_5]^{\oplus 4}$:

$$
\begin{aligned}
s - t_1 &= (1, 1, c, c), \\
C_1^{(1)} - C_1^{(2)} &= (1 - \omega_5)(1, -1, 0, 0), \\
C_1^{(2)} - C_1^{(3)} &= (1 - \omega_5)(0, 1, -(1 + \omega_5^3), 0), \\
C_1^{(3)} - C_1^{(4)} &= (1 - \omega_5)(0, 0, (1 + \omega_5^3), -(1 + \omega_5^3))
\end{aligned}
$$

where $c = \omega_5(2\omega_5^2 - \omega_5 + 2)$. A basis for the lattice L_5 is then

$$
\begin{aligned}
&l_1 = (1, 1, c, c) & &l_2 = \omega_5 l_1 \\
&l_3 = \omega_5^2 l_1 & &l_4 = \omega_5^3 l_1 \\
&l_5 = (1 - \omega_5)^2(0, 0, 2 + 4\omega_5 + \omega_5^2 + 3\omega_5^3, 0) & &l_6 = (1 - \omega_5)^2(1, 0, 0, 0) \\
&l_7 = \omega_5 l_6 & &l_8 = \omega_5^2 l_6 \\
&l_9 = (1 - \omega_5)(1, -1, 0, 0) & &l_{10} = (1 - \omega_5)^2(0, 1, 0, 0) \\
&l_{11} = \omega_5 l_{10} & &l_{12} = \omega_5^2 l_{10} \\
&l_{13} = (1 - \omega_5)(0, 1, -(1 + \omega_5^3), 0) & &l_{14} = (1 - \omega_5)^2(0, 0, 1, 0) \\
&l_{15} = \omega_5 l_{14} & &l_{16} = \omega_5^2 l_{14}.
\end{aligned}
$$

The identification between Ω_5 and L_5 is given by the map $b_i \mapsto l_i$. After this identification the intersection form on Ω_5 is exactly the form $b_{|L_5}$ on L_5. $\qquad\square$

Remark. 1) We recall that the density of a lattice L of rank n is $\Delta = V_n / \sqrt{\det L}$ where V_n is the volume of the n dimensional sphere of radius r (called *packing radius* of the lattice), $V_n = r^n \pi^{n/2}/(n/2)!$, $r = \sqrt{\mu}/2$ and μ is the minimal norm of a vector of the lattice.
The density of Ω_5 is $\Delta = \frac{\pi^8}{8!} \frac{1}{5^2} \approx 0.0094$.
2) The lattice Ω_5 does not admit vectors of norm -2 and can be generated by vectors of norm -4, and a basis is b_1, b_2, b_3, b_4, $b_5 - b_{13} - 2b_{14} - 3b_{15} - 4b_{16}$, b_6, b_7, b_8, b_9, $b_{10} + b_{11}$, $b_{11} + b_{12}$, $b_{10} + b_{11} + b_{12}$, b_{13}, $b_{14} + b_{15}$, $b_{15} + b_{16}$, $b_{14} + b_{15} + b_{16}$.

4.6. **Section of order seven.** Let X be a K3 surface with an elliptic fibration which admits a section of order seven as described in section 3. We recall that X has three reducible fibres of type I_7 and three singular irreducible fibres of type I_1. We have seen that the rank of the Néron-Severi group is 20. We determine now $NS(X)$ and T_X.
We label the fibers and their components as described in the section 3. Let t_1 denote the section of order seven which meets the first fiber in $C_1^{(1)}$. Again by the formula (4) of section 3 we have

$$
t_1 \cdot C_1^{(1)} = 1, \ t_1 \cdot C_2^{(2)} = 1, \ t_1 \cdot C_3^{(3)} = 1, \ \text{and } t_1 \cdot C_i^{(j)} = 0 \text{ otherwise.}
$$

Let σ_7 denote the automorphism of order seven which leaves each fiber invariant and is the translation by t_1, so $\sigma_7^*(s) = t_1$, $\sigma_7^*(t_1) = t_2$, $\sigma_7^*(t_2) = t_3$, $\sigma_7^*(t_3) = t_4$, $\sigma_7^*(t_4) = t_5$, $\sigma_7^*(t_5) = t_6$, $\sigma_7^*(t_6) = s$. The proofs of the next two propositions are very similar to those of the similar propositions in the case of the automorphisms of order three and five, so we omit them.

Proposition 4.5. *A $\mathbb{Z}$-basis for the lattice $NS(X)$ is given by*

$$
s, t_1, t_2, t_3, t_4, t_5, t_6, F, C_1^{(1)}, C_2^{(1)}, C_3^{(1)}, C_4^{(1)}, C_5^{(1)}, C_6^{(1)}, C_1^{(2)}, C_2^{(2)}, C_3^{(2)}, C_4^{(2)}, C_5^{(2)}, C_6^{(2)}.
$$

Let $U \oplus A_6^3$ be the lattice generated by the section, the fiber and the irreducible components of the three fibers I_7 which do not intersect the zero section s. It has index seven in the

Néron-Severi group of X, $NS(X)$.
The lattice $NS(X)$ has discriminant -7 and its discriminant form is $\mathbb{Z}_7(-\frac{4}{7})$.
The transcendental lattice T_X is the lattice $\{\mathbb{Z}^{\oplus 2}, \Upsilon\}$ where

$$\Upsilon := \begin{pmatrix} 4 & 1 \\ 1 & 2 \end{pmatrix}$$

and it has a unique primitive embedding in the lattice Λ_{K3}.

4.6.1. *The invariant lattice and its orthogonal complement.*

Proposition 4.6. *The invariant sublattice of the Néron-Severi lattice is isometric to the lattice $U(7)$ and it is generated by the classes F and $s + t_1 + t_2 + t_3 + t_4 + t_5 + t_6$.*
The invariant lattice $H^2(X, \mathbb{Z})^{\sigma_7^*}$ is isometric to $U(7) \oplus T_X$. Its orthogonal complement $\Omega_7 := (H^2(X, \mathbb{Z})^{\sigma_7^*})^\perp$ is the negative definite eighteen dimensional lattice $\{\mathbb{Z}^{18}, M\}$ where M is the bilinear form

$$\begin{pmatrix}
-4 & 2 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\
2 & -4 & 2 & 0 & 0 & 0 & 0 & 2 & -1 & 0 & 0 & 0 & 0 & -1 & 1 & 1 & -1 & 0 \\
0 & 2 & -4 & 2 & 0 & 0 & 7 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 1 \\
0 & 0 & 2 & -4 & 2 & 0 & -7 & 0 & -1 & 2 & -1 & 0 & 1 & -1 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 2 & -4 & 2 & 0 & 0 & 0 & -1 & 2 & -1 & -1 & 1 & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & -4 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & -1 & 1 & 1 & -1 \\
0 & 0 & 7 & -7 & 0 & 0 & -98 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 7 & -21 \\
-1 & 2 & -1 & 0 & 0 & 0 & 0 & -6 & 4 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 2 & -1 & 0 & 0 & 0 & 4 & -6 & 4 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 2 & -1 & 0 & 0 & -1 & 4 & -6 & 4 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & -1 & 4 & -6 & 4 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 2 & 0 & 0 & 0 & -1 & 4 & -6 & 3 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & -1 & 3 & -4 & 3 & -1 & 0 & 0 & 0 \\
1 & -1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & -6 & 4 & -1 & 0 & 0 \\
-1 & 1 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 4 & -6 & 4 & -1 & 0 \\
0 & 1 & -1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 4 & -6 & 4 & -1 \\
0 & -1 & 1 & 0 & 0 & 1 & 7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 4 & -6 & 4 \\
0 & 0 & 1 & -1 & 0 & -1 & -21 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 4 & -6
\end{pmatrix}$$

and it is equal to the lattice $(NS(X)^{\sigma_7^*})^\perp$.
The lattice Ω_7 admits a unique primitive embedding in the lattice Λ_{K3}.
The discriminant of Ω_7 is 7^3 and its discriminant form is $(\mathbb{Z}_7(\frac{4}{7}))^{\oplus 3}$.
The isometry σ_7^* acts on the discriminant group $\Omega_7^\vee/\Omega_7$ as the identity.

The basis of $(NS(X)^{\sigma_7^*})^\perp$ associated to the matrix M is $b_1 = s - t_1$, $b_2 = t_1 - t_2$, $b_3 = t_2 - t_3$, $b_4 = t_3 - t_4$, $b_5 = t_4 - t_5$, $b_6 = t_5 - t_6$ $b_7 = F - 7C_6^{(2)}$, $b_i = C_{i-7}^{(1)} - C_{i-6}^{(1)}$, $i = 8, \ldots, 12$, $b_{13} = C_1^{(1)} - C_1^{(2)}$, $b_i = C_{i-13}^{(2)} - C_{i-8}^{(2)}$, $i = 14, \ldots, 18$.

4.7. **The lattice Ω_7.** Let ω_7 be a primitive seventh root of the unity. In this section we prove the following result

Theorem 4.5. *The lattice Ω_7 is isometric to the $\mathbb{Z}$-lattice associated to the $\mathbb{Z}[\omega_7]$-lattice $\{L_7, h_{L_7}\}$ where*

$$L_7 = \left\{ (x_1, x_2, x_3) \in (\mathbb{Z}[\omega_7])^{\oplus 3} : \begin{array}{l} x_1 \equiv x_2 \equiv 6x_3 \quad \mod (1 - \omega_7), \\[2mm] (1 + 5\omega_7)x_1 + 3x_2 + 2x_3 \equiv 0 \quad \mod (1 - \omega_7)^2 \end{array} \right\}$$

with the hermitian form

$$(15) \qquad\qquad h_{L_7}(\alpha, \beta) = \alpha_1 \bar{\beta}_1 + f_1 \alpha_2 \overline{f_1 \beta_2} + f_2 \alpha_3 \overline{f_2 \beta_3},$$

where $f_1 = 3 + 2(\omega_7 + \omega_7^6) + (\omega_7^2 + \omega_7^5)$, $f_2 = 2 + (\omega_7 + \omega_7^6)$.

Proof. As in the previous cases we define the lattice $F := F_7^3$. We consider the hermitian form

$$h(\alpha, \beta) = \alpha_1 \bar{\beta}_1 + f_1 \alpha_2 \overline{f_1 \beta_2} + f_2 \alpha_3 \overline{f_2 \beta_3}$$

on the lattice $\mathbb{Z}[\omega_7]^{\oplus 3}$, and define G to be the sublattice $G = (1 - \omega_7)^2 \mathbb{Z}[\omega_7]^{\oplus 3}$ of $\{\mathbb{Z}[\omega_7]^{\oplus 3}, h\}$.
The map $\phi : F \to G$

$$\phi : \begin{array}{ccc} (\sigma_7^*)^i (C_1^{(1)} - C_2^{(1)}) & \mapsto & (1 - \omega_7)^2 \omega_7^i (1, 0, 0) \\ (\sigma_7^*)^i (C_1^{(2)} - C_2^{(2)}) & \mapsto & (1 - \omega_7)^2 \omega_7^i (0, 1, 0) \\ (\sigma_7^*)^i (C_1^{(3)} - C_2^{(3)}) & \mapsto & (1 - \omega_7)^2 \omega_7^i (0, 0, 1) \end{array}$$

is an isomorphism between the $\mathbb{Z}$-lattice $G_{\mathbb{Z}}$, with the bilinear form induced by the hermitian form, and F with the intersection form. We have to add to G some vectors to find a lattice L_7 isomorphic to Ω_7. These vectors are

$$\begin{array}{l} s - t_1 = (1, c, k), \\ C_1^{(1)} - C_1^{(2)} = (1 - \omega_7)(1, -(1 + \omega_7^4), 0), \\ C_1^{(2)} - C_1^{(3)} = (1 - \omega_7)(0, (1 + \omega_7^4), -(1 + \omega_7^3 + \omega_7^5)), \end{array}$$

where $c = 1 + 3\omega_7 + 3\omega_7^4 + \omega_7^5$ and $k = -5 + \omega_7 - 5\omega_7^2 - 3\omega_7^4 - 3\omega_7^5$. A basis for the lattice L_7 is

$$\begin{array}{ll} l_1 = (1, c, k) & l_2 = \omega_7 l_1 \\ l_3 = \omega_7^2 l_1 & l_4 = \omega_7^3 l_1 \\ l_5 = \omega_7^4 l_1 & l_6 = \omega_7^5 l_1 \\ l_7 = (1 - \omega_7)^2 (0, 2 + 4\omega_7 + 6\omega_7^2 + \omega_7^3 + 3\omega_7^4 + 5\omega_7^5, 0) & l_8 = (1 - \omega_7)^2 (1, 0, 0) \\ l_9 = \omega_7 l_8 & l_{10} = \omega_7^2 l_8 \\ l_{11} = \omega_7^3 l_8 & l_{12} = \omega_7^4 l_8 \\ l_{13} = (1 - \omega_7)(1, -(1 + \omega_7^4), 0) & l_{14} = (1 - \omega_7)^2 (0, 1, 0) \\ l_{15} = \omega_7 l_{14} & l_{16} = \omega_7^2 l_{14} \\ l_{17} = \omega_7^3 l_{14} & l_{18} = \omega_7^4 l_{14} \end{array}.$$

The identification between Ω_7 and L_7 is given by the map $b_i \mapsto l_i$. After this identification the intersection form on Ω_7 is exactly the form $b_{|L_7}$ on L_7 induced by the hermitian form (15). $\square$

Remark. 1) The density of Ω_7 is $\Delta = \frac{\pi^9}{9!} \frac{1}{\sqrt{7^3}} \approx 0.0044$.
2) As in the previous cases the lattice Ω_7 does not admit vectors of norm -2 and can be generated by vectors of norm -4, and a basis is b_1, b_2, b_3, b_4, b_5, b_6, $b_7 - b_{13} - 2b_{14} - 3b_{15} - 4b_{16} - 5b_{17} - 6b_{18}$, $b_8 + b_9$, $b_9 + b_{10}$, $b_{10} + b_{11}$, $b_{11} + b_{12}$, $b_{10} + b_{11} + b_{12}$, b_{13}, $b_{14} + b_{15}$, $b_{15} + b_{16}$, $b_{16} + b_{17}$, $b_{17} + b_{18}$, $b_{16} + b_{17} + b_{18}$.

5. FAMILIES OF K3 SURFACES WITH A SYMPLECTIC AUTOMORPHISM OF ORDER p

In the previous sections we used elliptic K3 surfaces to describe some properties of the automorphism σ_p. All these K3 surfaces have Picard number $\rho_p + 1$, where ρ_p is the minimal Picard number found in the Proposition 1.1. In this section we want to describe algebraic K3 surfaces with symplectic automorphism of order p and with the minimal possible Picard number. Recall that the values of ρ_p are

$$\begin{array}{cccc} p & 3 & 5 & 7 \\ \rho_p & 13 & 17 & 19, \end{array}$$

and Ω_p denote the lattices described in the sections 4.1, 4.4, 4.6.

Proposition 5.1. *Let X be a K3 surface with symplectic automorphism of order $p = 3, 5, 7$ and Picard number ρ_p as above. Let L be a generator of $\Omega_p^\perp \subset NS(X)$, with $L^2 = 2d > 0$ and let*

$$\mathcal{L}_{2d}^p := \mathbb{Z}L \oplus \Omega_p.$$

Then we may assume that L is ample and
(1) if $L^2 \equiv 2, 4, \ldots, 2(p-1) \mod 2p$, then $\mathcal{L}_{2d}^p = NS(X)$,
(2) if $L^2 \equiv 0 \mod 2p$, then either $\mathcal{L}_{2d}^p = NS(X)$ or $NS(X) = \widetilde{\mathcal{L}_{2d}^p}$ with $\widetilde{\mathcal{L}_{2d}^p}/\mathcal{L}_{2d}^p \simeq \mathbb{Z}/p\mathbb{Z}$ and in particular $\widetilde{\mathcal{L}_{2d}^p}$ is generated by an element $(L/p, v/p)$ with $v^2 \equiv 0 \mod 2p$ and $L^2 + v^2 \equiv 0 \mod 2p^2$.

Proof. Since $L^2 > 0$ by Riemann Roch theorem we can assume L or $-L$ effective. Hence we assume L effective. Let N be an effective (-2) curve then $N = \alpha L + v'$, with $v' \in \Omega_p$ and $\alpha > 0$ since Ω_p do not contains (-2)-curves. We have $L \cdot N = \alpha L^2 > 0$, and so L is ample. Moreover recall that L and Ω_p are primitive sublattices of $NS(X)$. Since the discriminant group of $\mathcal{L}_{2d}^p := \mathbb{Z}L \oplus \Omega_p$ is $(\mathbb{Z}/2d\mathbb{Z}) \oplus (\mathbb{Z}/p\mathbb{Z})^{\oplus n_p}$, with $n_3 = 6$, $n_5 = 4$, $n_7 = 3$ an element in $NS(X)$ not in $\mathcal{L}_{2d}^p$ is of the form $(\alpha L/2d, v/p)$, $v \in \Omega_p$ and satisfy the following conditions:
(a) $p \cdot (\alpha L/2d, v/p) \in NS(X)$,
(b) $(\alpha L/2d, v/p) \cdot L \in \mathbb{Z}$,
(c) $(\alpha L/2d, v/p)^2 \in \mathbb{Z}$.
By using the condition (a) we obtain $p \cdot (\alpha L/2d, v/p) - (0, v) \in NS(X)$ and so

$$\frac{p\alpha L}{2d} \in NS(X).$$

Hence by the primitivity of L in $NS(X)$ follows that $d \equiv 0 \mod p$, $d = pd'$, $d' \in \mathbb{Z}_{>0}$ and so

$$\frac{\alpha L}{2d'} \in NS(X)$$

which gives $\alpha = 2d'$ and the class (if there is) is $(L/p, v/p)$. Now condition (b) gives

$$(L/p, v/p) \cdot L = L^2/p \in \mathbb{Z}$$

and so $L^2 = 2p \cdot r$, $r \in \mathbb{Z}_{>0}$, since the lattice is even. And so if $NS(X) = \widetilde{\mathcal{L}_{2d}^p}$, then $L^2 \equiv 0 \mod 2p$. Finally condition (c) gives

$$(L/p, v/p)^2 = \frac{L^2 + v^2}{p^2}$$

and so since a square is even $L^2 + v^2 \equiv 0 \mod 2p^2$. $\square$

In the sections 4.1, 4.4, 4.6 we defined a symplectic automorphism σ_p, $p = 3, 5, 7$ of order p on some special K3 surfaces and we found the lattices $\Omega_p = (\Lambda_{K3}^{\sigma_p^*})^\perp$. Now we consider more in general an isometry on Λ_{K3} defined as σ_p^* (we call it again σ_p^*). In the next theorem we prove that if X is a K3 surface such that $NS(X) = \mathcal{L}_{2d}^p$ or $NS(X) = \widetilde{\mathcal{L}_{2d}^p}$, then this isometry is induced by a symplectic automorphism of the surface X.

Proposition 5.2. *Let $\mathcal{L}_p = \mathcal{L}_{2d}^p$ or $\mathcal{L}_p = \widetilde{\mathcal{L}_{2d}^p}$ if $p = 3, 5$ and let $\mathcal{L}_7 = \widetilde{\mathcal{L}_{2d}^7}$. Then there exists a K3 surface X with symplectic automorphism σ_p of order p such that $NS(X) = \mathcal{L}_p$ ($p = 3, 5, 7$) and $(H^2(X, \mathbb{Z})^{\sigma_p^*})^\perp = \Omega_p$.*
Moreover there are no K3 surfaces with Néron-Severi group isometric to $\mathcal{L}_{14d}^7$.

Proof. Let σ_p^*, $p = 3, 5, 7$, be an isometry as in the sections 4.1, 4.4, 4.6. We make the proof in several steps.
Step 1: there exists a marked K3 surface X such that $NS(X)$ is isometric to $\mathcal{L}_p$, and there are no K3 surfaces with Néron-Severi group isometric to $\mathcal{L}_{14d}^7$. By [Ni2, Theorem 1.14.4] the lattices $\mathcal{L}_{2d}^p$ $\widetilde{\mathcal{L}_{2d}^3}$, $\widetilde{\mathcal{L}_{2d}^5}$ have a unique primitive embedding in the K3 lattice. The lattice $T_5 = U(5) \oplus U(5) \oplus \langle -2d \rangle$ has a unique primitive embedding in Λ_{K3}, again by [Ni2, Theorem 1.14.4]. Its signature is $(2, 3)$ and its discriminant form is the opposite of the discriminant form of $\mathcal{L}_{2d}^5$. Since, by [Ni2, Corollary 1.13.3], $\mathcal{L}_{2d}^5$ is uniquely determined by its signature and discriminant form, it is the orthogonal of T_5 in Λ_{K3} and then $\mathcal{L}_{2d}^5$ admits a primitive embedding in Λ_{K3}. The lattice $\widetilde{\mathcal{L}_{2d}^7}$ is a primitive sublattice of the Néron-Severi group of the K3 surface described in the section 4.6, so it is a primitive sublattice of Λ_{K3} (the same argument can be applied to the lattices $\widetilde{\mathcal{L}_{2d}^p}$, $p = 3, 5$). Let now $\omega \in \mathcal{L}_p^\perp \otimes \mathbb{C} \subseteq \Lambda_{K3} \otimes \mathbb{C}$, with $\omega\omega = 0$, $\omega\bar{\omega} > 0$. We choose ω generic with these properties. By the surjectivity of the period map of K3 surfaces, ω is the period of a K3 surface X with $NS(X) = \omega^\perp \cap \Lambda_{K3} = \mathcal{L}_p$.
The rank of the lattice $\mathcal{L}_{14d}^7$ is 19 and its discriminant group has four generators. If $\mathcal{L}_{14d}^7$ was the Néron-Severi group of a K3 surface, the transcendental lattice of this surface should be a rank three lattice with a discriminant group generated by four elements. This is clearly impossible.
Step 2: the isometry σ_p^* fixes the sublattice $\mathcal{L}_p$. Since $\sigma_p^*(\Omega_p) = \Omega_p$ and $\sigma_p^*(L) = L$ (because $L \in \Omega_p^\perp$ which is the invariant sublattice of Λ_{K3}), if $\mathcal{L}_p = \mathcal{L}_{2d}^p = \mathbb{Z}L \oplus \Omega_p$ it is clear that $\sigma_p^*(\mathcal{L}_p) = \mathcal{L}_p$. Now we consider the case $\mathcal{L}_p = \widetilde{\mathcal{L}_{2d}^p}$. The isometry σ_p^* acts trivially on $\Omega_p^\vee / \Omega_p$ (cf. Propositions 4.2, 4.4, 4.6) and on $(\mathbb{Z}L)^\vee / \mathbb{Z}L$. Let $\frac{1}{p}(L, v') \in \mathcal{L}_p$, with $v' \in \Omega_p$. This is also an element in $(\Omega_p \oplus L\mathbb{Z})^\vee / (\Omega_p \oplus L\mathbb{Z})$. So we have $\sigma_p^*(\frac{1}{p}(L, v')) \equiv \frac{1}{p}(L, v')$ mod $(\Omega_p \oplus \mathbb{Z}L)$, which means

$$\sigma_p^*(\frac{1}{p}(L, v')) = \frac{1}{p}(L, v') + (\beta L, v''), \qquad \beta \in \mathbb{Z}, \quad v'' \in \Omega_p.$$

Hence we have $\sigma_p^*(\mathcal{L}_p) = \mathcal{L}_p$.
Step 3: The isometry σ_p^* is induced by an automorphism of the surface X. The isometry σ_p^* fixes the sublattice $\mathcal{L}_p^\perp$ of Λ_{K3}, so it is an Hodge isometry. By the Torelli theorem an effective Hodge isometry of the lattice Λ_{K3} is induced by an automorphism of the K3 surface (cf. [BPV, Theorem 11.1]). To apply this theorem we have to prove that σ_p^* is an effective isometry. An effective isometry on a surface X is an isometry which preserves the set of effective divisors. By [BPV, Corollary 3.11] σ_p^* preserves the set of the effective divisors if and only if it preserves the ample cone. So if σ_p^* preserves the ample

cone it is induced by an automorphism of the surface. This automorphism is symplectic by construction (it is the identity on the transcendental lattice $T_X \subset \Omega_p^\perp$), and so if σ_p^* preserves the ample cone, the theorem is proven.

Step 4: The isometry σ_p^* preserves the ample cone $\mathcal{A}_X$. Let $\mathcal{C}_X^+$ be one of the two connected components of the set $\{x \in H^{1,1}(X, \mathbb{R}) \mid (x,x) > 0\}$. The ample cone of a K3 surface X can be described as the set $\mathcal{A}_X = \{x \in \mathcal{C}_X^+ \mid (x,d) > 0 \text{ for each } d \text{ such that } (d,d) = -2, \ d \text{ effective}\}$. First we prove that σ_p^* fixes the set of the effective (-2)-curves. Since there are no (-2)-curves in Ω_p, if $N \in \mathcal{L}_p$ has $N^2 = -2$ then $N = \frac{1}{p}(aL, v')$, $v' \in \Omega_p$, for an integer $a \neq 0$. Since $\frac{1}{p}aL^2 = L \cdot N > 0$, because L and N are effective divisor, we obtain $a > 0$. The curve $N' = \sigma_p^*(N)$ is a (-2)-curve because σ_p^* is an isometry, hence N' or $-N'$ is effective. Since $N' = \sigma_p^*(N) = (aL, \sigma_p^*(v'))$ we have $-N' \cdot L = -aL^2 < 0$ and so $-N'$ is not effective. Using the fact that σ_p^* has finite order it is clear that σ_p^* fixes the set of the effective (-2)-curves.

Now let $x \in \mathcal{A}_X$ then $\sigma_p^*(x) \in \mathcal{A}_X$, in fact $(\sigma_p^*(x), \sigma_p^*(x)) = (x,x) > 0$ and for each effective (-2)-curve d there exists an effective (-2)-curve d' with $d = \sigma_p^*(d')$, so we have $(\sigma_p^*(x), d) = (\sigma_p^*(x), \sigma_p^*(d')) = (x, d') > 0$. Hence σ_p^* preserves $\mathcal{A}_X$ as claimed. $\qquad\square$

Corollary 5.1. *The coarse moduli space of $\mathcal{L}_p$-polarized K3 surfaces (cf. [Do] for the definition) $p = 3, 5, 7$ has dimension seven, three, respectively one and is a quotient of*

$$\mathcal{D}_{\mathcal{L}_p} = \{\omega \in \mathbb{P}(\mathcal{L}_p^\perp \otimes_\mathbb{Z} \mathbb{C}) : \ \omega^2 = 0, \ \omega\bar\omega > 0\}$$

by an arithmetic group $O(\mathcal{L}_p)$.

Remark. In particular the moduli space of K3 surfaces admitting a symplectic automorphism of order $p = 3, 5, 7$ has dimension respectively seven, three and one.

6. Final remarks

1. In Proposition 5.2 it would be interesting to prove the unicity of the lattices $\widetilde{\mathcal{L}_{2d}^p}$, this requires some careful analysis of the automorphism group of the lattices Ω_p, $p = 3, 5, 7$.

2. It is not difficult to give examples of K3 surfaces (not elliptic) in some projective space with a symplectic automorphism of order three or five. Consider for example the surfaces of $\mathbb{P}^3$:

$S1:$ $\quad q_4(x_0, x_1) + q_2(x_0, x_1)x_2x_3 + l_1(x_0, x_1)x_2^3 + l_1'(x_0, x_1)x_3^3 + ax_2^2x_3^2 = 0$

$S2:$ $\quad a_{01}x_0^2x_1^2 + a_{23}x_2^2x_3^2 + a_{0123}x_0x_1x_2x_3 + a_{02}x_0^3x_2 + a_{13}x_1^3x_3 + a_{12}x_1x_2^3 + a_{03}x_0x_3^3 = 0.$

where q_i is homogeneous of degree i, l_1, l_1' are linear forms, and $a_{ij} \in \mathbb{C}$. The surfaces S_1 resp. S_2 admit symplectic automorphisms of order three resp. of order five induced by the automorphisms of $\mathbb{P}^3$ given by $\sigma_3 : (x_0 : x_1 : x_2 : x_3) \longrightarrow (x_0 : x_1 : \omega_3 x_2 : \omega_3^2 x_3)$ and $\sigma_5 : (x_0 : x_1 : x_2 : x_3) \longrightarrow (\omega_5 x_0 : \omega_5^4 x_1 : \omega_5^2 x_2 : \omega_5^3 x_3)$. The automorphisms of $\mathbb{P}^3$ commuting with σ_3 resp. σ_5 form a space of dimension six, resp. four, since the equations depend on 13, resp. seven parameters the dimension of the moduli space is seven, resp. three as expected (this is the minimal possible dimension). In a similar way one can costruct many more examples. In the case of order seven automorphisms it is more difficult to give such examples. Already in the case of a polarization $L^2 = 2$, the K3 surface is the minimal resolution of the double covering of $\mathbb{P}_2$ ramified on a sextic with singular points and these are the fixed points of the automorphisms. One should resolve the singularities and analyze the action on the resolution before doing the double cover.

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